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The Application of Discrete Variational Identity on Semi-Direct Sums of Lie Algebras

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Abstract With non-semisimple Lie algebras, the trace identity was generalized to discrete spectral problems. Then the corresponding discrete variational identity was used to a class of semi-direct sums of Lie algebras in a lattice hierarchy case and obtained Hamiltonian structures for the associated integrable couplings of the lattice hierarchy. It is a powerful tool for exploring Hamiltonian structures of discrete soliton equations.

Keywords Semi-direct sum of Lie algebra · Discrete variational identity · Integrable coupling · Hamiltonian structure · Bilinear form

1 Introduction

The theory of integrable Hamiltonian systems with infinite dimensions has gone through rapid development since the late 1960s. In 1989, professor Tu proposed an efficient approach to searching for integrable Hamiltonian hierarchy [1]. And using Tu scheme, a lot of integrable systems with Physics significance have been obtained [2–4].

Recently,an algebraic approach to integrable coupling [5] was presented based on the concept of semi-direct sums of Lie algebras [6, 7], which provide a powerful tool to analyze integrable equations and integrable couplings.

Now we assume that a pair of matrix discrete spectral problems

$$\begin{cases} E\Phi = U\Phi = U(u,\lambda)\Phi \\ \Phi_t = V\Phi = V(u, Eu, E^{-1}u, \dots, \lambda)\Phi, \end{cases} \tag{1}$$

where $u = u(n,t)$ is the potential, Φ_t denotes the derivative with respect to t, λ is a spectral parameter, E is the shift operator. Then through the discrete zero curvature equation

$$U_t = (EV)U - UV \tag{2}$$

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Equation (1) can generate a discrete soliton equation

$$u_t = K = K(n, t, u, Eu, E^{-1}u, \ldots) \tag{3}$$

In order to generate integrable couplings of (3), we take a semi-direct sum of G with another matrix loop algebra G_c as follows [7]:

$$\overline{G} = G \oplus G_c \tag{4}$$

where G, G_c satisfy $[G, G_c] = \{[A, B] \mid A \in G, B \in G_c\}$. Obviously, G_c is an ideal Lie sub-algebra of $\overline{G}$.

We choose a pair of enlarged matrix discrete spectral problems

$$\begin{cases} E\overline{\Phi} = \overline{U}\,\overline{\Phi} = \overline{U}(\overline{u}, \lambda)\overline{\Phi}, \\ \overline{\Phi}_t = \overline{V}\,\overline{\Phi} = \overline{V}(\overline{u}, E\overline{u}, E^{-1}\overline{u}, \ldots, \lambda)\overline{\Phi} \end{cases} \tag{5}$$

where $\overline{U} = U + U_c, \overline{V} = V + V_c, U_c, V_c \in G_c$.

Then the corresponding enlarged discrete zero curvature equation

$$\overline{U}_t = (E\overline{V})\,\overline{U} - \overline{V}\,\overline{U} \tag{6}$$

is equivalent to

$$\begin{cases} U_t = (EV)U - VU, \\ U_{c,t} = [(EV)U_c - U_c V] + [(EV_c)U - UV_c] + [(EV_c)U_c - U_c V_c] \end{cases} \tag{7}$$

Thus, the whole system above provides a coupling system for (3).

2 Discrete Variational Identity and a Formula for the Constant γ

For a given spectral matrix $U = U(u, \lambda) \in G$, where G is a matrix loop algebra, we can define $rank(U)$, the rank function satisfies

$$rank(AB) = rank(A) + rank(B) \tag{8}$$

In order to keep the rank balance in equations,we define

$$rank(E) = rank(U) = 0 \tag{9}$$

the requirement $rank(E) = 0$ is due to the stationary discrete zero curvature equation

$$(EV)(EU) = UV \tag{10}$$

and the requirement $rank(U) = 0$ is due to the discrete spectral problem $E\Phi = U\Phi$ in (1).

Let the two arbitrary solutions V_1 and V_2 of (10) with the same rank be linearly related by

$$V_1 = \gamma V_2, \quad \gamma = const. \tag{11}$$

Since semi-direct sums of Lie algebras are not semisimple,we can replace the Killing forms with general bilinear forms to establish Hamiltonian structures for discrete soliton equations associated with semi-direct sums of Lie algebras.

Given a non-degenerate bilinear form $\langle .,. \rangle$ on G, with the symmetric property

$$\langle A, B\rangle = \langle B, A\rangle \tag{12}$$

and the invariance property under the multiplication

$$\langle A, BC\rangle = \langle AB, C\rangle \tag{13}$$

Consider a functional

$$\omega = \sum_{n\in Z}(\langle V, U_\lambda\rangle + \langle \Lambda, (EV)(EU) - UV\rangle) \tag{14}$$

which has the following variation constraint conditions:

$$\nabla_V \omega = U_\lambda + U(E^{-1}\Lambda) - \Lambda U, \qquad \nabla_\Lambda \omega = (EV)(EU) - UV \tag{15}$$

Theorem 1 [8] (The discrete variational identity under general bilinear forms)
Let G be a matrix loop algebra, and $U = U(u,\lambda) \in G$ be homogeneous in rank such that (10) has a unique solution $V \in G$ of a fixed rank up to a constant multiplier. Then for any solution $V \in G$ of (10), being homogeneous in rank, and any non-degenerate bilinear form $\langle .,. \rangle$ on G with the symmetric property (12) and the invariance property under the multiplication (13), we have the following discrete variational identity

$$\frac{\delta}{\delta u}\sum_{n\in Z}\langle V, U_\lambda\rangle = \lambda^{-r}\frac{\partial}{\partial\lambda}\lambda^{r}\left\langle V, \frac{\partial U}{\partial u}\right\rangle \tag{16}$$

where $\frac{\delta}{\delta u}$ is the variational derivative with respect to the potential u and γ is a constant.

Theorem 2 [8] *Let V be a solution to (10) and $\Gamma = VU$. If $\langle\Gamma,\Gamma\rangle \neq 0$, then the constant γ in the discrete variational identity (16) is given by*

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle\Gamma,\Gamma\rangle|. \tag{17}$$

3 The Hamiltonian Lattice System and Its Integrable Coupling

The discrete spectral problem for the lattice hierarchy is given by

$$E\phi = U(u,\lambda)\phi, \qquad U = U(v,p,\lambda) = \begin{pmatrix} 0 & -\frac{1}{r} \\ -r & \lambda + s \end{pmatrix}, \qquad \Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \tag{18}$$

Set $\Gamma = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) = \sum_{i\geq 0}\left(\begin{smallmatrix} a_i & b_i \\ c_i & -a_i \end{smallmatrix}\right)\lambda^{-i}$, the discrete stationary zero curvature equation $(E\Gamma)U - U\Gamma = 0$ gives rise to

$$\begin{cases} rb^{(1)} + \frac{1}{r}c = 0, \\ -\frac{1}{r}(a^{(1)} + a) + \lambda b^{(1)} + sb^{(1)} = 0, \\ -r(a^{(1)} + a) - \lambda c - sc = 0, \\ -\frac{1}{r}c^{(1)} - rb - \lambda(a^{(1)} - a) - s(a^{(1)} - a) = 0. \end{cases} \tag{19}$$

Set $a=\sum_{n\geq 0} a_n\lambda^{-n}$, $b=\sum_{n\geq 0} b_n\lambda^{-n}$, $c=\sum_{n\geq 0} c_n\lambda^{-n}$, then (19) becomes

$$\begin{cases} rb_n^{(1)}+\frac{1}{r}c_n=0,\\ -\frac{1}{r}(a^{(1)}+a)+\lambda b_{n+1}^{(1)}+sb_n^{(1)}=0,\\ -r(a_n^{(1)}+a_n)-c_{n+1}-sc_n=0,\\ -\frac{1}{r}c_n^{(1)}-rb_n-(a_{n+1}^{(1)}-a_{n+1})-s(a_n^{(1)}-a_n)=0. \end{cases} \tag{20}$$

This system can uniquely determine all sets of functions a_i, b_i and c_i, the first three sets are

$$\begin{cases} a_0=-\frac{1}{2}, b_0=c_0=0,\\ a_1=0, b_1=-\frac{1}{r^{-1}}, c_1=r,\\ a_2=-\frac{r}{r^{-1}}, b_2=\frac{s}{r}, c_2=-rs \end{cases}$$

The compatibility conditions of the matrix discrete spectral problems

$$E\Phi=U\Phi,\qquad \Phi_t=V^{[n]}\Phi,\qquad V^{[n]}=(\lambda^{n+1}\Gamma)_+ \tag{21}$$

where $(\lambda^{n+1}\Gamma)_+$ denotes the polynomial part of $\lambda^{n+1}\Gamma$ in λ.

Then determine the lattice hierarchy of soliton equations

$$\begin{cases} r_{t\ n}=c_{n+1},\\ s_{t\ n}=a_{n+1}^{(1)}-a_{n+1} \end{cases} \tag{22}$$

Since

$$\begin{aligned} \langle V,U_\lambda\rangle&=\mathrm{tr}(VU_\lambda)=\frac{1}{r}c,\\ \langle V,U_s\rangle&=\mathrm{tr}(VU_s)=\frac{1}{r}c,\\ \langle V,U_r\rangle&=\mathrm{tr}(VU_r)=\frac{2a}{r}+\frac{(\lambda+s)c}{r^2}=\frac{a-a^{(1)}}{r}, \end{aligned} \tag{23}$$

where $V=\Gamma U^{-1}$, by using the trace identity with $\gamma=0$, we obtain the Hamiltonian structures for the Toda lattice hierarchy:

$$u_{tn}=\begin{pmatrix} r\\ s\end{pmatrix}_{nt}=J_1\frac{\delta H_n}{\delta u},\qquad J_1=\begin{pmatrix} 0 & r\\ -r & 0\end{pmatrix},\qquad H_n=\sum_{n\in Z}\left(-\frac{c_{n+1}}{(n+1)r}\right),\quad n>0 \tag{24}$$

Now we introduce two Lie algebra of 4×4 matrix [7]

$$\overline{G}=G\oplus G_c=\left\{\begin{pmatrix} A_0 & 0\\ 0 & A_0\end{pmatrix}|A_0=\begin{pmatrix} a_1 & a_2\\ a_3 & a_4\end{pmatrix}\right\}\oplus\left\{\begin{pmatrix} 0 & A_1\\ 0 & 0\end{pmatrix}|A_1=\begin{pmatrix} a_5 & a_6\\ a_7 & a_8\end{pmatrix}\right\} \tag{25}$$

where a_i, $1\leq i\leq 8$ are real constants.

Define the mapping

$$\sigma:\overline{G}\to R^8,\qquad A\mapsto(a_1,\dots,a_8)^T,\qquad A=\begin{pmatrix} a_1 & a_2 & a_5 & a_6\\ a_3 & a_4 & a_7 & a_8\\ 0 & 0 & a_1 & a_2\\ 0 & 0 & a_3 & a_4\end{pmatrix} \tag{26}$$

The corresponding Lie bracket [., .] on R^8 can be computed as follows

$$\begin{aligned}[a,b]^T &= (a_3b_2 - b_3a_2, a_2b_1 - b_1a_2 + a_4b_2 - b_4a_2, a_3b_4 - b_3a_4 + a_1b_3 - b_1a_3,\\ &\quad a_2b_3 - b_2a_3, a_7b_2 - b_7a_2 + a_3b_6 - b_3a_6, a_6b_3 - b_6a_3 + a_2b_7 - b_2a_7,\\ &\quad a_6b_1 - b_6a_1 + a_8b_2 - b_8a_2 + a_2b_5 - b_2a_5 + a_4b_6 - b_4a_6,\\ &\quad a_5b_3 - b_5a_3 + a_7b_4 - b_7a_4 + a_1b_7 - b_1a_7 + a_3b_8 - b_3a_8)\\ &= a^T R(b)\end{aligned}$$

where $a = (a_1, \dots, a_8)^T, b = (b_1, \dots, b_8)^T \in R^8$

$$R(b) = \begin{pmatrix} 0 & -b_2 & b_3 & 0 & 0 & -b_6 & b_7 & 0\\ -b_3 & b_1 - b_4 & 0 & b_3 & -b_7 & b_5 - b_8 & 0 & b_7\\ b_2 & 0 & b_4 - b_1 & -b_2 & b_6 & 0 & b_8 - b_5 & -b_6\\ 0 & b_2 & -b_3 & 0 & 0 & b_6 & -b_7 & 0\\ 0 & 0 & 0 & 0 & 0 & -b_2 & b_3 & 0\\ 0 & 0 & 0 & 0 & -b_3 & b_1 - b_4 & 0 & b_3\\ 0 & 0 & 0 & 0 & b_2 & 0 & b_4 - b_1 & -b_2\\ 0 & 0 & 0 & 0 & 0 & b_2 & -b_3 & 0 \end{pmatrix} \tag{27}$$

A bilinear form on R^8 can be defined by

$$\langle a, b\rangle = a^T F b \tag{28}$$

where F is a constant matrix $F = (f_{ij})_{8\times 8}$, satisfies

$$F^T = F, \quad R(b)F = -R(b)F^T \tag{29}$$

Solving the system yields

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{30}$$

For the above spectral problem,we define the corresponding enlarged spectral matrix as follows

$$\overline{U} = \overline{U}(\overline{u}, \lambda) = \begin{pmatrix} U & U_a\\ 0 & U \end{pmatrix} \in G \oplus G_c, \qquad U_a = U_a(u, q) = \begin{pmatrix} 0 & u_1\\ u_2 & u_3 \end{pmatrix} \tag{31}$$

where u_1, u_2, u_3 are two dependent variable and the enlarged potential $\overline{u}$ reads $\overline{u} = (r, s, u_1, u_2, u_3)^T$.

To solve the corresponding enlarged stationary discrete zero curvature equation

$$(E\overline{\Gamma})\overline{U} - \overline{U}\overline{\Gamma} = 0, \qquad \overline{\Gamma} = \begin{pmatrix} \Gamma & \Gamma_a\\ 0 & \Gamma \end{pmatrix}, \qquad \Gamma_a = \begin{pmatrix} e & f\\ g & -e \end{pmatrix}$$

i.e.

$$(E\overline{\Gamma})\overline{U} - U\Gamma_a + (E\Gamma)U_a - U_a\Gamma = 0 \tag{32}$$

Set $e = \sum_{n\geq 0} e_n\lambda^{-n}$, $f = \sum_{n\geq 0} f_n\lambda^{-n}$, $g = \sum_{n\geq 0} g_n\lambda^{-n}$ then (32) becomes

$$\begin{cases} rb_n^{(1)} + \frac{1}{r}c_n = 0, \\ -\frac{1}{r}(a_n^{(1)} + a_m) + b_{n+1}^{(1)} + sb_n^{(1)} = 0, \\ r(a_n^{(1)} + a_n) + sc_n + c_{n+1} = 0, \\ (a_{n+1}^{(1)} - a_{n+1}) + s(a_n^{(1)} - a_n) + rb_n + \frac{1}{r}c_n^{(1)} = 0, \\ u_2b_n^{(1)} + rf_n^{(1)} + \frac{1}{r}g_n - u_1c_n^{(1)} = 0, \\ u_1a_n^{(1)} + u_3b_n^{(1)} - \frac{1}{r}c_n^{(1)} + sf_n^{(1)} + f_{n+1}^{(1)} + \frac{1}{r}h_m + u_1a_m = 0, \\ -u_2a_n^{(1)} + rh_n^{(1)} - re_n - g_{n+1} - sg_n - u_2a_n - u_3c_n = 0, \\ u_1c_n^{(1)} - u_3a_n^{(1)} - \frac{1}{r}g_n^{(1)} + h_{n+1}^{(1)} + sh_n^{(1)} - rf_n - h_{n+1} - sh_n - u_2b_n + u_3a_n = 0 \end{cases} \tag{33}$$

and the initial values are: $a_0 = -\frac{1}{2}$, $b_0 = c_0 = e_0 = f_0 = g_0 = j_0 = a_1 = e_1 = h_1 = 0, b_1 = -\frac{1}{r^{(-1)}}$, $c_1 = r$, $f_1 = u$, $g_1 = u_2$, $a_2 = -\frac{r}{r^{(-1)}} \cdots$

Now we define

$$\overline{V}^{[n]} = \begin{pmatrix} V^{[n]} & V_a^{[n]} \\ 0 & V^{[n]} \end{pmatrix} \in \overline{G}, \qquad V_a^{[n]} = (\lambda^{n+1}\Gamma_a)_+,\ n \geq 0 \tag{34}$$

where $V^{[n]}$ is defined as in (21) and $(\lambda^{n+1}\Gamma_a)_+$ denotes the polynomial part of $\lambda^{n+1}\Gamma_a \in \lambda$.

Then the m-th enlarged discrete zero curvature equation

$$\overline{U}_{t\ n} = (E\overline{V}^{[n]})\overline{U} - \overline{U}\ \overline{V}^{[n]}$$

leads to

$$\begin{cases} r_{tn} = c_{n+1}, \\ s_{tn} = a_{n+1}^{(1)} - a_{n+1}, \\ (u_1)_{tn} = -f_{n+1}^{(1)}, \\ (u_2)_{tn} = g_{n+1}, \\ (u_3)_{tn} = h_{n+1} - h_{n+1}^{(1)} \end{cases} \tag{35}$$

i.e.

$$\begin{pmatrix} r \\ s \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}_{tn} = \begin{pmatrix} c_{n+1}, \\ a_{n+1}^{(1)} - a_{n+1}, \\ -f_{n+1}^{(1)}, \\ g_{n+1} \\ h_{n+1} - h_{n+1}^{(1)} \end{pmatrix} = J_2 \begin{pmatrix} \frac{h_{n+1}^{(1)} - h_{n+1}}{r} + \frac{a_{n+1} - a_{n+1}^{(1)}}{r} - u_1 a_{n+1}^{(1)} - \frac{u_2}{r^2}a_{n+1} \\ \frac{c_{n+1}}{r} - \frac{u_2}{r^2}c_{n+1} + \frac{g_{n+1}}{r} \\ -ra_{n+1}^{(1)} \\ \frac{a_{n+1}}{r} \\ \frac{c_{n+1}}{r} \end{pmatrix}$$

$$J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & r \\ 0 & 0 & -\frac{1}{r} & -r & 0 \\ 0 & \frac{1}{r} & 0 & 0 & -(\frac{1}{r} + u_1) \\ 0 & r & 0 & 0 & u_2 - r \\ -r & 0 & \frac{1}{r} + u_1 & r - u_2 & 0 \end{pmatrix} \tag{36}$$

To construct Hamiltonian structure of these integrable couplings by using the discrete variational identity (16) and (27), (29), direct computation we have

$$\langle V, \overline{U}_\lambda \rangle = \langle V, \overline{U}_s \rangle = cr^{-1} - u_2 r^{-2} + gr^{-1},$$

$$\langle V, \overline{U}_r \rangle = \left((\lambda + s)c + ra + u_3 c - \frac{\lambda + s}{r} u_2 c - r^2 u_1 b + (\lambda + s)g - rh + r(\lambda + s)u_1 c \right) r^{-2},$$

$$+ ar^{-1} - u_2 r^{-2} a + r^{-1} e,$$

$$\langle V, \overline{U}_{u_1} \rangle = (\lambda + s)c + ra,$$

$$\langle V, \overline{U}_{u_2} \rangle = ar^{-1},$$

$$\langle V, \overline{U}_{u_3} \rangle = cr^{-1},$$

where $\overline{U}$ is defined by (31) and $\overline{V} = \overline{\Gamma U}^{-1}$, $\overline{\Gamma}$ being defined by (32).

Then,by using the discrete variational identity (16) with $\gamma = 0$, we get

$$\frac{\delta}{\delta \overline{u}} \overline{H}_n = \left(\frac{h_{n+1}^{(1)} - h_{n+1}}{r} + \frac{a_{n+1} - a_{n+1}^{(1)}}{r} - u_1 a_{n+1}^{(1)} - \frac{u_2}{r^2} a_{n+1}, \frac{c_{n+1}}{r} - \frac{u_2}{r^2} c_{n+1} + \frac{g_{n+1}}{r}, \right.$$
$$\left. - r a_{n+1}^{(1)}, \frac{a_{n+1}}{r}, \frac{c_{m+1}}{r} \right)^T \tag{37}$$

$$\overline{H}_{n+1} = \sum_{n \in Z} \left(\frac{u_2 c_{n+2} - r c_{n+2} - r g_{n+2}}{(n+1) r^2} \right), \quad n \geq 0$$

Consequently, we obtain the Hamiltonian structure for the hierarchy of integrable couplings in (34):

$$\overline{u}_{tn} = \begin{pmatrix} v \\ p \\ u \\ q \end{pmatrix}_{tn} = J_2 \frac{\delta}{\delta \overline{u}} \overline{H}_{n+1}, \quad n > 0 \tag{38}$$

where the Hamiltonian functional $\overline{H}_{n+1}, n \geq 0$ are given in (37) and the Hamiltonian operator J_2 is defined in (36).

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